Groundstates of $SU(2)$-Symmetric Confined Bose Gas: Trap for a Schrödinger Cat

A.B. Kuklov[1] and B.V. Svistunov[2]
[1] *The College of Staten Island, CUNY, Staten Island, New York 10314*
[2] *Russian Research Center "Kurchatov Institute", 123182 Moscow, Russia*

Conservation of the total isotopic spin S of a two-component Bose gas — like ^{87}Rb — has a dramatic impact on the structure of the ground state. In the case when S is much smaller than the total number of particles N, the condensation of each of the two components occurs into at least *two* single-particle modes. The quantum wavefunction of such a groundstate is a Schrödinger Cat — a superposition of the *phase separated* classical condensates, the most "probable" state in the superposition corresponding to the classical groundstate in the sector of given N and S.

PACS numbers: 03.75.Fi

Symmetry has profound implications on the ground-state structure of the degenerate superfluids and superconducting systems. A remarkable example to that is superfluid ^{3}He, which exhibits a variety of textures [1]. These ground states are essentially *classical* — quantum fluctuations play no role at large length scales. Accordingly, the most important low energy physics can be well described within the mean-field (MF) approach relying on broken gauge symmetry.

Recent successes in trapping and manipulating atomic Bose-Einstein condensate (BEC) [2,3] have initiated a great interest to the physics beyond the MF in these systems. A well-known example of the non-MF behavior is the phase diffusion effect [4].

A remarkable case of non-MF ground state was found in the spin $s = 1$ sodium vapor [5,6]. This state is characterized by anomalous quantum fluctuations of the components with different projections of the total spin.

In this Letter, we address the case of a two-component $SU(2)$-symmetric Bose gas—like ^{87}Rb [7], that is, the case of (pseudo-)spin-1/2 bosons. We show that, while sharing some common features with the $s = 1$ bosons [5,6], the $s = 1/2$ system exhibits unique properties [8]. We pose a question of the formation of the ground state in a naturally arising situation of $S \ll N$. We find that the resulting ground state is built on at least *two* macroscopically populated lowest one-particle modes, and is a *quantum superposition of the phase separated BECs*, characterized by large fluctuations of the densities of the components, the total density being fixed.

We start from the $SU(2)$-symmetric Hamiltonian

$$H = \int d\mathbf{x}\, [\Psi^\dagger_\sigma H_1 \Psi_\sigma + \frac{g}{2}\Psi^\dagger_\sigma \Psi^\dagger_{\sigma'} \Psi_{\sigma'} \Psi_\sigma]\,, \qquad (1)$$

where the interaction constant $g = 4\pi a/m$ is represented (in units $\hbar = 1$) by the scattering length a and mass m; the single particle Hamiltonian H_1 consists of the kinetic energy and the trapping potential, $H_1 = -(1/2m)\nabla^2 + U(\mathbf{x})$; the Bose-field operators Ψ_σ have two components $\sigma =\uparrow, \downarrow$ in pseudo-spin notations, and the summation is performed over repeated pseudo-spin (Greek) indices. Due to its symmetry, this Hamiltonian conserves the total isospin $\mathbf{S} = (S_x, S_y, S_z)$ given as ($S_\pm = S_x \pm iS_y$, $S_+ \equiv S^\dagger_-$)

$$S_z = \frac{1}{2}\int d\mathbf{x}\, (\Psi^\dagger_\uparrow \Psi_\uparrow - \Psi^\dagger_\downarrow \Psi_\downarrow)\,, \quad S_- = \int d\mathbf{x}\, \Psi^\dagger_\downarrow \Psi_\uparrow\,. \qquad (2)$$

These operators obey the standard $SU(2)$ algebra $[S_z, S_+] = S_+$, $[S_z, S_-] = -S_-$, $[S_+, S_-] = 2S_z$.

The absolute minimum of energy is achieved when all atoms condense to a lowest one-particle (Hartree) state φ_0. Such a state corresponds to the maximum possible value of the isospin $S_{\max} = N/2 = (N_\uparrow + N_\downarrow)/2$, where $N_\uparrow$, $N_\downarrow$ stand for the total numbers of atoms of each component. Thus, it follows that a very nontrivial situation will occur when, while conserving S, the condensation proceeds from the thermal cloud which contains almost the same amount of different components N_σ with *no inter-component correlations*. Indeed, the ensemble mean of the total isospin (2) in Boltzmann gas is $\langle \mathbf{S}\rangle = 0$, and the mean of $\mathbf{S}^2 = S_z^2 + (S_+S_- + S_-S_+)/2$ yields $\langle \mathbf{S}^2\rangle \sim N$, as long as $\langle \Psi^\dagger_\uparrow(\mathbf{x})\Psi_\downarrow(\mathbf{x'})\rangle = 0$ and the diagonal correlators decay on thermal length. Similarly, $\langle \mathbf{S}^2\rangle \sim N$, when two spatially separated degenerate components are mixed together (see below). Thus, the resulting ensemble of the ground states is characterized by typical values $0 \leq S \sim \sqrt{N}$.

Since condensation of both components into one single-particle state φ_0 is incompatible with $S \ll N$, the resulting many-body state is characterized by macroscopic occupation of at least *two* lowest one-particle states. This contrasts the cases $s = 1/2$ and $s = 1$ [5,6].

We analyze the simplest case of the strong quasi-1D trap, where the energy difference $\varepsilon = \varepsilon_1 - \varepsilon_0$ between the lowest single particle levels $\varphi_{1,0}$, respectively, in the trapping potential U is much larger than a typical interaction energy per particle μ (chemical potential at $T = 0$). The inequality $\mu/\varepsilon \ll 1$ allows one to essentially simplify the description by introducing a projected Hamiltonian acting in a truncated Hilbert space containing only two single-particle modes, φ_0 and φ_1:

$$\Psi_\sigma = \sum_{b=0,1} a_{b\sigma}\varphi_b \qquad (3)$$

where $a_{b\sigma}$ annihilates a particle at the level b in the spin state σ. With the same accuracy, one may neglect in the projected Hamiltonian all the terms that do not conserve the total occupations of each of the two single-particle levels. These terms involve a large energy difference $\sim \varepsilon \gg \mu$, and, thus, result only in higher-order—in parameter μ/ε—perturbative renormalization of the parameters of the effective (two-modes & two-colors) Hamiltonian

$$H' = \sum_{a=0,1} \varepsilon_a N_a + \\ + \frac{1}{2} \sum_{a,b=0,1} I_{ab} N_a (N_b - \delta_{a,b}) + 2 I_{01} \mathbf{S}_0 \mathbf{S}_1, \qquad (4)$$

where $I_{ab} = g \int d\mathbf{x}\, |\varphi_a|^2 |\varphi_b|^2$ and $N_b = \sum_\sigma N_{b\sigma}$, $S_{bz} = (N_{b\uparrow} - N_{b\downarrow})/2$, $S_{bx} = (a^\dagger_{b\uparrow} a_{b\downarrow} + a^\dagger_{b\downarrow} a_{b\uparrow})/2$, $S_{by} = (a^\dagger_{b\uparrow} a_{b\downarrow} - a^\dagger_{b\downarrow} a_{b\uparrow})/2i$; the operators $N_{a\sigma} = a^\dagger_{b\sigma} a_{b\sigma}$ and $\mathbf{S}_a = (S_{ax}, S_{ay}, S_{az})$ represent the numbers of atoms and the components of the pseudo-spin on the ath level, respectively. The quantities N_a and $\mathbf{S}^2_a$ are related to each other by the identity $\mathbf{S}^2_a = N_a(1 + N_a/2)/2$, implying that $S_a \equiv N_a/2$.

A standard choice of eigen numbers for the Hamiltonian (4) comes from the theory of two spins: the total spin S, its z-projection, M, and the spins S_0 and S_1; with the constraints $|S_0 - S_1| \le S \le S_0 + S_1$, $-S \le M \le S$. In this nomenclature, the spectrum of (4) is

$$E_{SMS_0S_1} = 2\varepsilon_0(S_0 + S_1) + 2\varepsilon S_1 + \sum_{a,b} I_{ab} S_a (2S_b - \delta_{a,b}) \\ + I_{01}[S(S+1) - (S_0+1)S_0 - (S_1+1)S_1]. \qquad (5)$$

The distribution over the eigennumbers S_0, S_1, S, M, which will be formed in the end of the cooling processes, depends essentially on the following: While S, M and N *are* the eigennumbers of H, eq.(1), the spins S_0, S_1 *are not* the eigennumbers of the original Hamiltonian (1). Thus, the formation of particular low energy states, amenable to treatment by the reduced Hamiltonian (4) where $S_{0,1}$ are now the quantum numbers, will be controlled by the microcanonical distribution with fixed values of S, M, N. For our purposes, we do not need to adopt any particular distribution for S, M and N. We assume that their values are obtained (after the cooling is over) by a non-destructive measurement. Thus, the only remaining degree of freedom, within the chosen microcanonical ensemble, is the integer $0 \le K \le 2S$, that parameterizes $S_1 = (K - S + N/2)/2$ and $S_0 = (-K + S + N/2)/2$ in eq.(5). The corresponding Gibbs distribution over K with the temperature created by the cooling follows from the spectrum eq.(5), which now takes a form $E_{SMN}(K) = (\varepsilon_0 + \varepsilon/2)N + \varepsilon(K - S) + o(\mu/\varepsilon)$ in the main $\varepsilon \gg \mu$ limit. It indicates that the ground state for given S, M, N corresponds to $K = 0$. The next excited $K = 1$ state within the microcanonical ensemble costs energy ε. Thus, once temperature $T < \varepsilon$, the microcanonical ensemble for given S, M, N essentially consists of only one state—the ground state $K = 0$, given as

$$|SMN\rangle = Z_{SMN} \left(a^\dagger_{0\uparrow}\right)^{S+M} \left(a^\dagger_{0\downarrow}\right)^{S-M} \left(R^\dagger\right)^{N/2-S} |0\rangle, \qquad (6)$$
$$R = a_{0\uparrow} a_{1\downarrow} - a_{1\uparrow} a_{0\downarrow},$$

where $S_0 = (S + N/2)/2$ and $S_1 = (-S + N/2)/2$; Z_{SMN} is the normalization coefficient, and $|0\rangle$ is the vacuum. These states realize the irreducible representation of the isotopic $SU(2)$ group. For $S = 0, M = 0$ (N even), the state (6) is the $SU(2)$-singlet.

The state (6) exhibits quantum fluctuations of individual values of $N_{a\sigma}$, preserving, however, their three linear combinations: $N_{0\uparrow} + N_{0\downarrow} = S + N/2$, $N_{1\uparrow} + N_{1\downarrow} = -S + N/2$, $N_{1\uparrow} + N_{0\uparrow} - N_{1\downarrow} - N_{0\downarrow} = 2M$. Thus, there is only *one* degree of freedom, which we choose to be $N_{0\uparrow}$. Most transparently, these fluctuations are described in the basis of the Fock states $|N_{0\uparrow}, N_{0\downarrow}, N_{1\uparrow}, N_{1\downarrow}\rangle$, which are the *fragmented condensates* [9]. Expanding (6) over the Fock basis, we get the probability to find a given value of $N_{0\uparrow}$:

$$P_{SMN}(N_{0\uparrow}) = \frac{Z^2_{SMN} N_{0\uparrow}!(N/2 + S - N_{0\uparrow})!}{(N_{0\uparrow} - S - M)!(N/2 + M - N_{0\uparrow})!}. \qquad (7)$$

To estimate the strength of the quantum fluctuations, consider $\overline{\delta N^2_{0\uparrow}}$, where for any Q: $\overline{Q} = \langle SMN|Q|SMN\rangle$ and $\overline{\delta Q^2} = \langle SMN|(Q - \overline{Q})^2|SMN\rangle$. At $S = 0$, we have $P_{SMN}(N_{0\uparrow}) = 2/N$ over the whole range of $0 \le N_{0\uparrow} \le N/2$; and $\overline{\delta N^2_{0\uparrow}} = N^2/12$. For $1 \ll S \ll N$, $S - |M| \gg 1$, the Gaussian approximation of (7) gives

$$\overline{N}_{0\uparrow} \approx (S + M)(S + N/2)/2S\,, \qquad (8)$$
$$\overline{\delta N^2_{0\uparrow}} \approx (S^2 - M^2)((N/2)^2 - S^2)/8S^3. \qquad (9)$$

For typical values $S \sim \sqrt{N}$, we obtain $\overline{\delta N^2_{0\uparrow}} \approx N^2/(8S) \sim N^{3/2}$. In a direct analogy with the $s = 1$ case [5], these anomalous fluctuations are consequence of the conservation of the total (iso)spin [6].

The state (6) is a quantum superposition — a Schrödinger Cat — of the *phase separated* "classical" BEC's, each BEC being characterized by well-defined spatial density distribution of the spin-up and spin-down components. The phase separation is due to the involvement of two single-particle modes with different occupations of the spin-up/down components.

Measurement of $N_{0\uparrow}$ produces a *collapse* of the Cat. The resulting states essentially depend on how strong the interaction between the measuring device and the system is. The most "accurate" measurement of $N_{0\uparrow}$, with the absolute error less than one particle, would collapse the Cat into a Fock state $|N_{0\uparrow}, N_{0\downarrow}, N_{1\uparrow}, N_{1\downarrow}\rangle$, with the probability (7) to obtain given $N_{0\uparrow}$. Such a measurement, however, implies that the interaction of the system

with the apparatus is much stronger than that due to the Hamiltonian (4). In this sense, such a measurement is destructive not only to the supposed-to-be-fragile Quantum Cat, but also to its classical counterparts.

More interesting is a less destructive measurement, which collapses Quantum Cat into some "classical" (in a particular sense) states. These classical states are the phase-coherent BEC states, which can be interpreted in terms of the classical vectors $\mathbf{S}_0, \mathbf{S}_1$ interacting antiferromagnetically by the term $2I_{01}\mathbf{S}_0 \cdot \mathbf{S}_1 = I_{01}[S^2 - S_1^2 - S_0^2]$, $I_{01} > 0$. Comparing the classical energy with the term in the square brackets in the full (quantum) expression (5), we see that the creation of such classical states costs only the energy $\delta E_C \approx I_{01}[S_0 + S_1] \sim I_{01}N \sim \mu$. Here we took into account that $S_0 \sim S_1 \sim N$ for $S \ll N$.

The above-discussed classical BEC states are readily obtained within the Gross-Pitaevskii approach. As an instructive example, let us construct the classical groundstates. To this end we replace the bosonic fields operators by the c-fields. Accordingly, in eqs. (3,4), we replace the a-operators by the c-number amplitudes, and find the minimum of energy at given N, S, M. This yields $a_{0\uparrow} = \sqrt{S + N/2}\cos(\beta/2)\exp(i(\alpha_0 + \gamma)/2)$, $a_{0\downarrow} = -\sqrt{S + N/2}\sin(\beta/2)\exp(i(\alpha_0 - \gamma)/2)$, $a_{1\uparrow} = i\sqrt{-S + N/2}\sin(\beta/2)\exp(i(\alpha_1 + \gamma)/2)$, $a_{1\downarrow} = i\sqrt{-S + N/2}\cos(\beta/2)\exp(i(\alpha_1 - \gamma)/2)$. Here $\cos\beta = M/S$, $(\alpha_1 - \alpha_0)/2 = -\varepsilon(1 + o(\mu/\varepsilon))t$; α_0 is the global phase, and γ determines the x, y-projections of the total spin ($S_x = -S\sin\beta\cos\gamma$). Comparison of these with eq.(8) shows that the classical occupation numbers coincide with the expectations for the quantum ones.

The classical fields are non-stationary, which is a consequence of the energy difference between the states φ_0 and φ_1. Hence, wherever $\varphi_0(\mathbf{x})$ and $\varphi_1(\mathbf{x})$ have substantial overlapping, the spatial densities of the two components experience strong fluctuations, the total density remaining constant. These fluctuations are of purely classical nature and have nothing to do with the above-discussed quantum fluctuations of occupation numbers! It is reasonable to reveal these classical fluctuations directly in the quantum wavefunction (6). To this end we calculate fluctuations of the difference of the component densities, $\eta(\mathbf{x}) = \Psi^\dagger_\uparrow(\mathbf{x})\Psi_\uparrow(\mathbf{x}) - \Psi^\dagger_\downarrow(\mathbf{x})\Psi_\downarrow(\mathbf{x})$ [Note that $\overline{\eta} \approx 0$ for $|M| \ll S$, that is the state (6) *does not* exhibit any phase separation *on average*]. For the fluctuations of η we get

$$\overline{\delta\eta^2} = 4[\varphi_0^4 + \varphi_1^4 - 4\varphi_0^2\varphi_1^2]\overline{\delta N_{0\uparrow}^2} + 2\varphi_0^2\varphi_1^2[(N/4)(N+4) - \\ - S(S+1) + 4\overline{S_{0z}}(M - \overline{S_{0z}})] \,. \tag{10}$$

The two terms in (10) correspond to the quantum ($\sim \overline{\delta N_{0\uparrow}^2}$) and classical ($\sim N^2$ for $S \ll N$) fluctuations of η, respectively. Though the second term is dominant in a general case, it vanishes in the trap center, where $\varphi_1 = 0$, and the fluctuations are of purely quantum nature.

To propose a particular set of measurements that could reveal the statistics (7), we note that the quantities $N_0 = \sum_\sigma N_{0\sigma} = N/2 + S$, $N_1 = \sum_\sigma N_{1\sigma} = N/2 - S$, $N_\uparrow = N_{0\uparrow} + N_{1\uparrow} = N/2 + M$, $N_\downarrow = N_{0\downarrow} + N_{1\downarrow} = N/2 - M$ *do not* fluctuate. Thus, the values S, M, N can be extracted, in effect, *without* destroying the wave function. Once the set of values S, M, N is fixed, one can measure $N_{0\uparrow}$, by, e.g., observing the density of the spin-up component in the center of the trap by the spatially differentiated selective imaging [10]. In the trap center $\varphi_1 = 0$, therefore the component densities are unambiguously related to the occupations of the state φ_0 [11]. Actual measurements always have some uncertainties ΔM, ΔS, ΔN in determining M, S, N, respectively. These may wash out the quantum fluctuations. First, we note that the mean value $\overline{N}_{0\uparrow}$ in eq.(8) is exactly the value given within the classical geometrical picture. Thus, $\langle(\Delta\overline{N}_{0\uparrow})^2\rangle$, where $\langle ... \rangle$ denotes the statistical mean, must be significantly smaller than the quantum fluctuation $\overline{\delta N_{0\uparrow}^2}$, eq.(9). Practically, for $\langle(\Delta N)^2\rangle \le N$, $\langle S\rangle \sim \sqrt{N}$, we find that, as long as $\langle(\Delta M)^2\rangle \ll \langle S\rangle/2$, the quantum fluctuations can be well distinguished.

We consider an intriguing alternative to the direct cooling of the two-component system in order to obtain the state (6). Namely, the *merging* of two spatially separated condensed species from two identical wells with equal numbers of oppositely "polarized" particles in each of them. At $T \ll \varepsilon$ (which means, for our purposes, practically zero), and in the absence of tunneling between the wells, the system is in its lowest energy state. We note further that our state is $(N/2 + 1)$-fold degenerate within the given sector of N and $M = 0$, the other $N/2$ groundstates being obtained by exchanging spin-up–spin-down pairs between the wells. The key point is that the initial state is a superposition of the eigenstates of the total pseudo-spin operator with the typical values $0 \le S \lesssim \sqrt{N}$, and the degeneracy is exhausted by the isospin index (which, at $M = 0$, has exactly $N/2 + 1$ values, from 0 to $N/2$). Thus, each of the S-eigenstates in our superposition is *not degenerate*. Accordingly, when tunneling is slowly turned on and, finally, the inter-well barrier is adiabatically removed [12], the system remains in the superposition of the ground S-eigenstates, which at $\varepsilon/\mu \gg 1$ take on the form (6) ($M = 0$). Projection on the particular S can be done by, say, measuring total density in the trap center [11].

Other (more exotic) options for the direct Cat-state engineering rely on merging more than two BEC clouds in the presence of the controlled $SU(2)$ symmetry violating potential. We will analyze them elsewhere.

Let us discuss the effect of the *symmetry breaking static perturbation*, which was shown to destroy the quantum fluctuating state in the case $s = 1$ in the thermodynamical limit [6]. A similar situation occurs in the

case of the pseudo-spin $s = 1/2$ bosons. A role of the perturbation is played by the gradient $\mathbf{B}'$ of the pseudo-magnetic field $\mathbf{B}' = (0, 0, B')$, consisting of the difference $U' = U_\uparrow - U_\downarrow$ of the trapping potentials for the corresponding components and of the interaction term $\tilde{H} = (g'/2)\int d\mathbf{x}\,[\Psi^\dagger_\uparrow\Psi^\dagger_\uparrow\Psi_\uparrow\Psi_\uparrow - \Psi^\dagger_\downarrow\Psi^\dagger_\downarrow\Psi_\downarrow\Psi_\downarrow]$, where $g' = 4\pi a'/m$ and $|a'| \ll a$ in ^{87}Rb [7]. Within the model, we find the corresponding term

$$H_B = -B'(S_{0z} - S_{1z}),$$
$$B' = \int d\mathbf{x}\,(U'(\mathbf{x}) + \frac{g'}{2}\rho(\mathbf{x}))(\varphi_1^2 - \varphi_0^2), \tag{11}$$

to be added to H', eq.(4), where we take into account $N_0 \approx N_1 \approx N/2$, and $\rho(\mathbf{x}) = N(\varphi_0^2 + \varphi_1^2)/2$.

The term H_B does not commute with the total spin, which implies that it *does not* favor quantum fluctuations. It decreases energy of the classical (not fluctuating) state $|BEC\rangle = (a^\dagger_{0\uparrow}a^\dagger_{1\downarrow})^{N/2}|0\rangle$ by $\delta E_{BEC} \approx -\langle BEC|H_B|BEC\rangle \approx B'N$, ($B' > 0$). At the same time, the classical states loose energy with respect to the lowest eigenstate of H' , eq.(4), by the chemical potential μ. Accordingly, in order to insure the stability of the Schrödinger Cat state (6), the condition $\mu \geq |B'|N$ should hold. This condition cannot be obviously satisfied in the thermodynamical limit, and should be considered in mesoscopic situations only. We note, however, that it can be well controlled within the current experimental capabilities. Taking the effective potential breaking the symmetry as the oscillator potential $\delta U = m\nu^2x^2/2$, with some frequency ν [13], we find $B' = \nu^2/(2\varepsilon)$ in eq.(11), where we have employed the lowest eigenstates $\varphi_{0,1}$, of the 1D oscillator with frequency ε. Accordingly, ν should obey $\nu/\varepsilon \leq \sqrt{\mu/(\varepsilon N)}$. For $N \leq 10^4$, $\mu \leq \omega_0$, this yields $\nu/\varepsilon \leq 10^{-2}$, which is well within the accuracy of the experiment [13]. In fact, the control of the value B' together with the external rf-pulses [7,10] provides a tool for manipulating the parameters of the Cat state.

Another effect of the symmetry breaking, existing even for $B' = 0$, is opening the channel for irreversible transitions $1 \to 0$. While bringing the system to the absolute ground state ($S = N/2$), it is, however, of a *second* order in g', and needs the normal component to be present to proceed at noticeable rate. In ^{87}Rb [7], $g/g' \approx 30$. Thus, a time scale for a such relaxation is a factor of 10^3 longer than relaxation time $\sim 1/g^2$ of any lowest energy excitation at considered low temperatures and densities, which makes it irrelevant for all practical purposes, if compared with the time of the Cat-state formation $\sim 1/\mu \sim 1/g$ [12].

Let us consider stability of the above analysis with respect to external *symmetry breaking temporal noise*. For example, the trapping potential may be slightly different for the components and can fluctuate in time. These fluctuations may produce two qualitatively different effects: 1) heating of the cloud at some rate τ_h^{-1}, and 2) disruption of the coherence at a rate τ_{irr}^{-1}, which, generally speaking, is different from τ_h^{-1}.

A most detrimental for the Cat situation may occur when the noise, while producing no significant heating, disrupts the anomalous quantum correlations $A(t) = \overline{\delta N_{0\uparrow}^2}$, as given by eq.(9), at times $\tau_{irr} \ll \tau_h$. As discussed in ref. [14], this occurs when the noise correlation length is larger than a typical thermal length of the cloud. Accordingly, if the noise is produced by small correlation length factors, such as, e.g., collisions with the background gas, the decoherence *is not* faster than the heating.

In order to investigate the case under consideration, we note that $A(t)$ can naturally be represented in terms of the one-particle $\rho^{(1)}(\mathbf{x}, \mathbf{x}', t) = \langle\Psi^\dagger(\mathbf{x}, t)\Psi(\mathbf{x}', t)\rangle$ (OPDM) and two-particle $\rho^{(2)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}'_2, \mathbf{x}'_1, t) = \langle\Psi^\dagger(\mathbf{x}_1, t)\Psi^\dagger(\mathbf{x}_2, t)\Psi(\mathbf{x}'_2, t)\Psi(\mathbf{x}'_1, t)\rangle$ (TPDM) density matrices, respectively, where the averaging is performed over the initial state, which is taken to be the Cat-state (6), and over the noise. Accordingly, the equations for them should be analyzed and the rates τ_{irr}^{-1}, τ_h^{-1} evaluated and, then, compared with each other. In the case of the white noise, this can be done, practically, exactly with the help of the Furutzu-Novikov theorem as discussed in ref. [14].

The heating rate is defined as the inverse time τ_h required to increase the total kinetic energy per particle $K(t) = -(1/m)\nabla^2_{\mathbf{x}=\mathbf{x}'}\rho^{(1)}(\mathbf{x}, \mathbf{x}', t)/N \sim t$ by ε. The decoherence rate τ_{irr}^{-1} is defined as the inverse of a typical *shortest* time of the decay of the elements of the OPDM and the TPDM. All rates can be evaluated with respect to the OPDM and the TPDM solutions in the limit $t \to 0$.

It is important to note that, while the heating is most effectively produced by a short range noise, the coherence is destroyed by the long range noise due to the long range temporal fluctuations of the trapping potential $U'(\mathbf{x}, t)$. These fluctuations can be produced by thermal or any other noise of electric currents $I'(t)$ in the magnetic coils. For the purpose of estimating, we assume that

$$U' = U(\mathbf{x})I'(t)/I, \;\; \langle I'(t)I'(t')\rangle = C^2\delta(t - t'), \tag{12}$$

where $U(\mathbf{x})$ is some typical trapping potential, which does not fluctuate; I stands for the static current; we assumed the white noise structure of the current fluctuations, which is characterized by the constant C^2. In the case of large temperatures of the coils T_C, this constant is given by the fluctuation-dissipation theorem as $C^2 = 2T_C/R$, where R stands for the resistance. Proceeding as described above and in ref. [14] and taking into account eq.(12), the heating and the decoherence rates can be found as

$$\tau_h^{-1} \approx \overline{(\nabla U(\mathbf{x}))^2}C^2/(m\varepsilon I^2), \;\; \tau_{irr}^{-1} \approx \overline{U^2(\mathbf{x})}C^2/I^2, \tag{13}$$

respectively, where the line stands for a typical average over the cloud volume, and the numerical coefficients of order 1-10 (depending on the order of the correlator) are omitted.

We estimate $\overline{U^2(\mathbf{x})} \approx \varepsilon^2$ and take into account that typical kinetic energy is of the order of the potential energy, that is, $\overline{(\nabla U(\mathbf{x}))^2}/m \approx \varepsilon^3$. For the thermal noise, where $C^2 = 2T_C/R$, $T_C = 300K$ (room temperature), $\varepsilon/\hbar \approx 10^3\mathrm{s}^{-1}$, we find $\tau_{irr}^{-1} \approx \varepsilon^2 k_B T/(\hbar^2 R I^2) \sim 10^{-15}\mathrm{s}^{-1}$, if the coil power $RI^2 \sim 1\mathrm{W}$ (we restored the standard units). Accordingly, the thermal noise of the coil currents can be ignored for all practical purposes.

In a general case of a non-thermal noise characterized by some relaxation time τ_C and the current fluctuation magnitude $\langle I^{'2}\rangle$, so that $\langle I'(t)I'(t')\rangle = \langle I^{'2}\rangle \exp(-|t-t'|/\tau_C)$, the white noise result [14] and the estimate (13) can still be employed as long as $\tau_C \ll \tau_{irr}$. We, then, take $C^2 \approx \tau_C \langle I^{'2}\rangle$, and obtain $\tau_{irr}^{-1} \approx (\varepsilon/\hbar)^2 \tau_C \langle I^{'2}\rangle / I^2$. This implies that, for , e.g., $\tau_C = 10^{-6}\mathrm{s}$, $\varepsilon/\hbar \approx 10^3\mathrm{s}^{-1}$, the relative fluctuations $\langle I^{'2}\rangle/I^2$ can be as large as 10% and yet $\tau_{irr} > 100\mathrm{s}$, which should be compared with the time of the Cat-state formation [12]. As discussed above, this time can be as short as $\sim 1/\mu \sim 10^{-2} - 10^{-3}\mathrm{s}$.

The decoherence due to losses can similarly be analyzed within the framework of equations for the OPDM and TPDM. Regardless of the nature of the interaction vertex responsible for the losses, these equations can be obtained as an expansion with respect to the vertex, and, then, the rate τ_{irr}^{-1} compared with the corresponding contributions to the rate of losses τ_{loss}^{-1}, defined as the time derivative of $\int d\mathbf{x}\rho^{(1)}(\mathbf{x},\mathbf{x},t)$ per one particle. Performing this general analysis, we find that τ_{irr}^{-1} and τ_{loss}^{-1} are of the same order. Thus, once the losses are kept low during the time of the experiment (the scale of which is determined by the adiabaticity requirement [12]), the decoherence can be ignored.

In summary, we have discussed the role of the isotopic $SU(2)$ symmetry in the formation of the ground states in a confined mesoscopic two-component Bose gas. The conservation of the total isotopic spin results in condensation of a thermal cloud, consisting of the two uncorrelated components, on *two* lowest single particle states in the trap. The resulting ground state is essentially nonclassical Schrödinger Cat state characterized by anomalously large non-uniform fluctuations of the density of each component about its classical expectation. The symmetry breaking term may destabilize this state into the phase separation, with both components becoming the classical BEC. This process, however, can be well controlled experimentally. The realization of the fluctuating many body state provides unique opportunity for observing the effect of measurement induced collapse of the Schrödinger Cat to classical states. The conditions required for distinguishing the quantum fluctuation from the statistical noise are delineated.

The authors are grateful to N.P. Bigelow, M.J. Holland, A.E. Meyerovich, N.V. Prokof'ev, and Li You for their interest to this work and useful discussions. ABK was supported by CUNY grant PSC-63499-0032. BVS acknowledges a support from Russian Foundation for Basic Research under from the Netherlands Organization for Scientific Research (NWO).

[1] G.E. Volovik, in *Helium Three*, Ed.: W.P. Halperin, L.P. Pitaevskii, Amsterdam-Tokyo (1990), p.27.

[2] M.H. Anderson, *et al.* Science **269**, 198 (1995); K.B. Davis, *et al.* Phys. Rev. Lett. **75**, 3969 (1995); C.C. Bradley, *et al.* Phys. Rev. Lett. **75**, 1687 (1995).

[3] D.M. Stamper-Kurn, *et al.* Phys. Rev. Lett. **80**, 2027 (1998).

[4] E.M. Wright, D.F. Walls, J.C. Garrison, Phys. Rev. Lett. **77**, 2158(1996); M. Lewenstein and L. You, Phys. Rev. Lett **77**, 3489 (1996); A.B. Kuklov, *et al.* Phys. Rev. A **55**, 488 (1997); L.P. Pitaevskii, Phys. Lett. A **229**, 406 (1997); Y. Castin, R. Dum, Phys. Rev. A **57**, 3008 (1998).

[5] C.K. Law, H. Pu, and N.P. Bigelow, Phys. Rev. Lett. **81**, 5257 (1998).

[6] T.-L. Ho and S.K. Yip, Phys. Rev. Lett. **84**, 4031 (2000).

[7] D.S. Hall, *et al.* Phys. Rev. Lett. **81**, 1539 (1998).

[8] A crutial circumstance that distinguishes $s = 1/2$ bosons from the $s = 1$ ones is the conservation of the number of particles of each spin component. Hence, apart from a special symmetry of the Hamiltonian, $s = 1/2$ bosons can be considered as a mixture of two components, each component possessing a well-defined number of particles.

[9] P. Noziéres, in *Bose-Einstein Condensation*, Ed. A. Griffin, D.W. Snoke, S. Stringari, Cambridge University Press, p. 15 (1995).

[10] M.R. Matthews, *et al.* Phys. Rev. Lett. **83**, 3358 (1999).

[11] For example, measurement of the total density in the center gives $(N/2+S)\varphi_0^2(0)$, which yields S. This should be contrasted with the density $N\varphi_0^2(0)$ in the absolute ground state $S = N/2$.

[12] Initially, within the given S-sector, the lowest excited state costs energy $\sim \mu$. Thus, the adiabaticity time scale must be $\gg 1/\mu$.

[13] H. J. Lewandowski, *et al.* cond-mat/0109476.

[14] A.B. Kuklov, et al., Laser Physics **12**, 127 (2002); cond-mat/0106611.